Equality Conditions for Lower Bounds on the Smallest Singular Value of a Bidiagonal Matrix

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Abstract

Several lower bounds have been proposed for the smallest singular value of a square matrix, such as Johnson's bound, Brauer-type bound, Li's bound and Ostrowski-type bound. In this paper, we focus on a bidiagonal matrix and investigate the equality conditions for these bounds. We show that the former three bounds give strictly lower bounds if all the bidiagonal elements are nonzero. For the Ostrowski-type bound, we give an easily verifiable necessary and sufficient condition for the equality to hold.

Key words: Singular values, Lower bounds, Equality conditions, Bidiagonal matrix, dqds algorithm.

1 Introduction

The singular values are fundamental quantities that describe the properties of a given matrix. In particular, the smallest singular value plays a special role in numerical linear algebra and several lower bounds for estimating it from below have been proposed so far. Examples of the lower bounds include Johnson's bound, Ostrowski-type bound, Brauer-type bound and Li's bound.

In a certain situation, we are interested to know whether equality holds in these lower bounds. For example, lower bounds can be used to determine the shifts in the dqds or related algorithms for singular value computation. In that case, to guarantee global convergence and numerical stability, we must make sure that the bound is strictly smaller than the smallest singular value.

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In this paper, we focus on a bidiagobnal matrix and study the equality conditions for the four lower bounds listed above. We show that if all the diagonal and upper subdiagonal elements of the bidiagonal matrix are nonzero, Johnson's bound, Brauer-type bound and Li's bound all give strict lower bounds. The restriction here is not serious since any bidiagonal matrix can be transformed easily to satisfy it. For Ostrowski-type bound, we give a necessary and sufficient condition for equality to hold.

In section 2, we review the four lower bounds for the smallest singular value. In section 3, we give two theorems concerning the equality conditions for these bounds. Section 4 gives an example of a bidiagonal matrix for which the Ostrowski-type bound gives the exact smallest singular value.

2 Lower bounds on the smallest singular value

We consider an n by n upper bidiagonal matrix B given by

$$B = \begin{pmatrix} b_{11} & -b_{12} & & \\ & b_{22} & \ddots & \\ & & \ddots & -b_{n-1,n} \\ & & & b_{nn} \end{pmatrix} \tag{1}$$

and denote its smallest singular value by $\sigma_n(B)$. Following [2], we assume that B has *Property (A)* defined below:

Definition 2.1. *An upper bidiagonal matrix B is said to have Property (A) if all the diagonal elements are positive and all the upper subdiagonal elements of are negative, i.e., $b_{ii} > 0$ $(i = 1, \ldots, n)$ and $b_{i,i+1} > 0$ $(i = 1, \ldots, n-1)$.*

If B has Property (A), the right and left singular vectors corresponding to $\sigma_n(B)$ can be chosen positive since they are the eigenvectors of positive matrices $(B^TB)^{-1}$ and $(BB^T)^{-1}$, respectively, corresponding to the largest eigenvalue $(\sigma_n(B))^{-2}$.

We can show that our assumption is not restrictive as follows. If one of the subdiagonal elements of B is zero, B is decomposed into a direct sum of two upper diagonal matrices. So we can compute the lower bounds for each matrix separately. If a diagonal element of B is zero, by applying one step of the dqds algorithm with zero shift, we can chase the zero element to the lower-right corner. By deflating the element, we obtain a smaller bidiagonal matrix with nonzero diagonals [2]. Finally, the diagonal elements and upper subdiagonal elements can be made positive and negative, respectively, by multiplying appropriate diagonal matrices with diagonal elements ± 1 from both sides.

Now we can state the four lower bounds that we will deal with in this paper. In the following, we adopt the convention that $b_{0,1} = b_{n,n+1} = 0$.

Theorem 2.2. *Johnson bound[6]:*

$$\sigma_n(B) \geq \min_{1 \leq k \leq n} \left\{ b_{kk} - \frac{1}{2} \left(b_{k-1,k} + b_{k,k+1} \right) \right\}. \tag{2}$$

Theorem 2.3. *Brauer-type bound[7]:*

$$\begin{aligned}\sigma_n(B) \geq \min_{1 \leq j < k \leq n} \frac{1}{2} \Big\{ & b_{kk} + b_{jj} \\ & - \sqrt{\left(b_{kk} - b_{jj} \right)^2 + \left(b_{k-1,k} + b_{k,k+1} \right) \left(b_{j-1,j} + b_{j,j+1} \right)} \Big\}.\end{aligned} \tag{3}$$

Theorem 2.4. *Li's bound[8]:*

$$\begin{aligned}\sigma_n(B) \geq \min_{1 \leq k \leq n-1} \frac{1}{2} \Big\{ & b_{kk} + b_{k+1,k+1} \\ & - \sqrt{\left(b_{kk} - b_{k+1,k+1} \right)^2 + \left(b_{k-1,k} + b_{k,k+1} \right) \left(b_{k,k+1} + |b_{k+1,k+2}| \right)} \Big\}.\end{aligned} \tag{4}$$

Theorem 2.5. *Ostrowski-type bound[7]:*

$$\sigma_n(B) \geq \min_{1 \leq k \leq n} \left\{ \sqrt{b_{kk}^2 + \frac{1}{4} \left(b_{k-1,k} - b_{k,k+1} \right)^2} - \frac{1}{2} \left(b_{k-1,k} + b_{k,k+1} \right) \right\}. \tag{5}$$

It can be shown that Li's bound is stronger than the Brauer-type bound, which in turn is stronger than Johnson's bound [7]. Also, Ostrowski's bound is stronger than the Johnson bound [7]. Brauer-type bound and Ostrowski-type bound are mtually noncomparable. Also, Li's bound and Ostrowski-type bound are mutually noncomparable.

3 Equality conditions for the lower bounds

We first study equality conditions in Johnson bound, Brauer-type bound and Li's bound. To this end, we show the following theorem. This theorem is stated in [4], p. 151, but here we restrict ourselves to a bidiagonal matrix and give a sufficient condition for the strict inequality to hold.

Lemma 3.1. *Let B be an n by n upper bidiagonal matrix. Also, for an n by n symmetric matrix A, denote its smallest eigenvalue by $\lambda_n(A)$. Then*

$$\sigma_n(B) \geq \lambda_n\left(\frac{1}{2}\left(B + B^T\right)\right). \tag{6}$$

Moreover, the strict inequality holds if B has Property (A).

Proof. Let $\lambda \equiv \lambda_n\left(\frac{1}{2}(B + B^T)\right)$ and $\sigma \equiv \sigma_n(B)$ and denote the normalized right and left singular vectors of B corresponding to σ by $\mathbf{x}$ and $\mathbf{y}$, respectively. Then,

$$(B^T - \lambda I)(B - \lambda I) = B^T B - \lambda(B + B^T) + \lambda^2 I \succeq O. \tag{7}$$

Here, $A \succeq B$ means that the matrix$A - B$ is positive semidefinite and O means the zero matrix. By multiplying $\mathbf{x}^T$ and $\mathbf{x}$ from both sides we have

$$\sigma^2 = \mathbf{x}^T B^T B \mathbf{x} \geq \lambda \mathbf{x}^T (B + B^T)\mathbf{x} - \lambda^2 \mathbf{x}^T \mathbf{x} \geq 2\lambda^2 - \lambda^2 = \lambda^2. \tag{8}$$

Since $\sigma \geq 0$, it follows that $\sigma \geq |\lambda| \geq \lambda$.

Next, we consider the case where $b_1, b_2, \ldots, b_{2n-1}$ are all nonzero and $\lambda > 0$ and show $\sigma > \lambda$ by contradiction. Assume $\sigma = \lambda$. Then, since the second inequality in (8) must be equality, we have

$$\lambda \mathbf{x}\left(\frac{1}{2}(B + B^T)\right)\mathbf{x} = \lambda^2. \tag{9}$$

Noting that B is nonsingular and $\sigma = \lambda \neq 0$, we get

$$\mathbf{x}\left(\frac{1}{2}(B + B^T)\right)\mathbf{x} = \lambda. \tag{10}$$

Since λ is the smallest eigenvalue of $\frac{1}{2}(B + B^T)$, Eq. (10) means that $\mathbf{x}$ is an eigenvector of $\frac{1}{2}(B + B^T)$ belonging to λ. Hence,

$$\frac{1}{2}(B + B^T)\mathbf{x} = \lambda \mathbf{x}. \tag{11}$$

By repeating the same argument with B and $\mathbf{x}$ replaced by B^T and $\mathbf{y}$, respectively, we have

$$\frac{1}{2}(B + B^T)\mathbf{y} = \lambda \mathbf{y}. \tag{12}$$

But, since $\frac{1}{2}(B+B^T)$ is a symmetric tridiagonal matrix whose subdiagonal elements are all nonzero, all of its eigenvalues are distinct and the corresponding eigenspaces are one-dimensional [9]. We can therefore conclude that $\mathbf{x} = \mathbf{y}.$ By combining this with $B\mathbf{x} = \sigma\mathbf{y}$ and $\sigma = \lambda$, we have $B\mathbf{x} = \lambda\mathbf{x}$. Also, $B^T\mathbf{x} = \lambda\mathbf{x}$ from $B^T\mathbf{y} = \sigma\mathbf{x}$.

Now, at least one of the diagonal elements of B must be equal to λ. Let $b_{kk} = \lambda$. Then, since B is a bidiagonal matrix whose subdiagonal elements are all nonzero, we can easily show that $x_{k+1} = x_{k+2} = \cdots = x_n = 0$ by comparing the components of $B\mathbf{x} = \lambda\mathbf{x}$. Also, By comparing the components of $B^T\mathbf{x} = \lambda\mathbf{x}$, we have $x_{k-1} = x_{k-1} = \cdots = x_1 = 0$. Thus $\mathbf{x} = \mathbf{e}_k$ (the k-th column of the identity matrix). However, if $k > 1$, we can insert this into the k-th component of $B\mathbf{x} = \lambda\mathbf{x}$:

$$b_{k-1,k-1}x_{k-1} + b_{k-1,k}x_k = \lambda x_{k-1}, \tag{13}$$

and obtain $b_{k-1,k} = 0$, which is a contradiction. If $k = 1$, from the second component of $B^T\mathbf{x} = \lambda\mathbf{x}$, we have $b_{12} = 0$, which also is a contradiction. Hence $\sigma > \lambda$ must hold when all the bidiagonal elements are nonzero. $\square$

Using Lemma 3.1, we can derive the following results.

Theorem 3.2. *Let B be an upper bidiagonal matrix that has Property (A). Then strict inequality holds for the Johnson bound, Brauer-type bound and Li's bound.*

Proof. Let $A = (a)_{ij}$ be an $n \times n$ symmetric tridiagonal matrix and denote its smallest eigenvalue by $\lambda_n(A)$. Then we have from Gerschgorin's theorem [3],

$$\lambda_n(A) \geq \min_{1\leq k\leq n} \left[a_{kk} - (|a_{k,k-1}| + |a_{k,k+1}|)\right]. \tag{14}$$

Similarly, from Brauer's theorem (Cassini's oval) [3] we have

$$\lambda_n(A) \geq \min_{1\leq j<k\leq n} \frac{1}{2}\Big[a_{kk} + a_{jj} - \sqrt{(a_{kk} - a_{jj})^2 + 4\left(|a_{k,k-1}| + |a_{k,k+1}|\right)\left(|a_{j,j-1}| + |a_{j,j+1}|\right)}\Big]. \tag{15}$$

Li showed that in the right hand side of eq. (15), we need to consider only those j and k which satisfy $(a_{jk}, a_{kj}) \neq (0, 0)$. Hence in the tridiagonal case, we have the following (refined) inequality:

$$\lambda_n \geq \min_{1\leq k\leq n-1} \frac{1}{2}\Big[a_{kk} + a_{k+1,k+1}$$
$$-\sqrt{(a_{kk} - a_{k+1,k+1})^2 + 4\left(|a_{k,k-1}| + |a_{k,k+1}|\right)\left(|a_{k+1,k}| + |a_{k+1,k+2}|\right)}\Big]. \tag{16}$$

By letting $A = \frac{1}{2}\left(B + B^T\right)$ and combining eqs. (14), (15) and (16) with Lemma 3.1, we can derive the Johnson bound, Brauer-type bound and Li's bound, respectively. It is clear from Lemma 3.1 that strict inequalitiy holds if B has Property (A). □

The situation is different for the Ostrowski-type bound. In fact, as the next theorem states, the Ostrowski-type lower bound can reach the smallest singular value, although in a very special case.

Theorem 3.3. *Let B be an upper bidiagonal matrix that has Property (A). Denote the normalized positive right and left singular vectors of B corresponding to $\sigma_n(B)$ by $\mathbf{x}$ and $\mathbf{y}$, respectively. Then the following three conditions are equivalent:*

(a) Equality holds in the Ostrowski-type lower bound.
(b) In the right-hand-side of the Ostrowski-type lower bound, the argument of the minimum gives the same positive value for all k.
(c) $y_k = x_{k+1}$ $(1 \leq k \leq n-1)$ and $y_n = x_1$.

Proof. We first derive the Ostrowski-type bound itself for completeness and then show (a) ⇒ (b), (b) ⇒ (c) and (c) ⇒ (a) in this order.

To begin with, we note that the matrix

$$A = \begin{pmatrix} B & -\sigma_n I \\ -\sigma_n I & B^T \end{pmatrix} \tag{17}$$

is singular since $A\,[\mathbf{x}^T, \mathbf{y}^T]^T = \mathbf{0}$. Then by applying Ostrowski's theorem for eigenvalue inclusion region [3]

$$\forall i, \exists k, \;\; |\lambda_i - a_{kk}| \leq \left(\sum_{i\neq k} |a_{ik}|\right)^{1/2} \left(\sum_{j\neq k} |a_{kj}|\right)^{1/2}, \tag{18}$$

to A and setting $\lambda_i = 0$, we have

$$\exists k, \; b_{kk} \leq \sqrt{b_{k-1,k} + \sigma_n}\sqrt{b_{k,k+1} + \sigma_n}. \tag{19}$$

Solving this for σ_n gives

$$\exists k,\ \sigma_n \geq \sqrt{b_{kk}^2 + \frac{1}{4}\left(b_{k-1,k} - b_{k,k+1}\right)^2} - \frac{1}{2}\left(b_{k-1,k} + b_{k,k+1}\right), \tag{20}$$

from which the Ostrowski-type bound (5) follows immediately.

(a) ⇒ (b): Assume that equality holds in eq. (5). Then,

$$\forall k,\ \sigma_n \leq \sqrt{b_{kk}^2 + \frac{1}{4}\left(b_{k-1,k} - b_{k,k+1}\right)^2} - \frac{1}{2}\left(b_{k-1,k} + b_{k,k+1}\right). \tag{21}$$

Since all the variables are positive, this is equivalent to

$$\forall k,\ b_{kk}^2 \geq (b_{k-1,k} + \sigma_n)(b_{k,k+1} + \sigma_n). \tag{22}$$

Hence we have

$$\forall k,\ \frac{b_{kk}^2}{b_{k,k+1} + \sigma_n} \geq b_{k-1,k} + \sigma_n \tag{23}$$

and

$$\forall k,\ \frac{b_{kk}^2}{b_{k-1,k} + \sigma_n} \geq b_{k,k+1} + \sigma_n. \tag{24}$$

By multiplying eq (23) and eq (24) with x_k^2 and y_k^2, respectively, and summing over k, we have

$$\begin{aligned} &\sum_{k=1}^{n} \left\{ \frac{b_{kk}^2}{b_{k,k+1} + \sigma_n} \cdot x_k^2 + \frac{b_{kk}^2}{b_{k-1,k} + \sigma_n} \cdot y_k^2 \right\} \\ &\geq \sum_{k=1}^{n} \left\{ (b_{k-1,k} + \sigma_n) x_k^2 + (b_{k,k+1} + \sigma_n) y_k^2 \right\}. \end{aligned} \tag{25}$$

On the other hand, from the k-th component of $B\mathbf{x} = \sigma_n \mathbf{y}$, we have

$$\begin{aligned} b_{kk} x_k &= \sigma_n y_k + b_{k,k+1} x_{k+1} \\ &= \sqrt{\sigma_n} \cdot \sqrt{\sigma_n} y_k + \sqrt{b_{k,k+1}} \cdot \sqrt{b_{k,k+1}} x_{k+1} \\ &\leq \sqrt{\sigma_n + b_{k,k+1}} \sqrt{\sigma_n y_k^2 + b_{k,k+1} x_{k+1}^2}. \end{aligned} \tag{26}$$

Here, we used Schwarz's inequality in the second inequality. Squaring both sides of eq. (26) gives

$$\frac{b_{kk}^2}{b_{k,k+1}+\sigma_n}\cdot x_k^2 \le \sigma_n y_k^2 + b_{k,k+1}x_{k+1}^2. \tag{27}$$

Similarly,

$$\frac{b_{kk}^2}{b_{k-1,k}+\sigma_n}\cdot y_k^2 \le \sigma_n x_k^2 + b_{k-1,k}y_{k-1}^2. \tag{28}$$

Summing eqs. (27) and (28) over all k, we have (noting that $b_{0,1} = b_{n,n+1} = 0$)

$$\begin{aligned}&\sum_{k=1}^{n}\left\{\frac{b_{kk}^2}{b_{k,k+1}+\sigma_n}\cdot x_k^2 + \frac{b_{kk}^2}{b_{k-1,k}+\sigma_n}\cdot y_k^2\right\}\\ &\le \sum_{k=1}^{n}\left\{(b_{k-1,k}+\sigma_n)x_k^2 + (b_{k,k+1}+\sigma_n)y_k^2\right\}.\end{aligned} \tag{29}$$

But eq. (25) shows that the opposite inequality holds. Thus the inequality in (25) and (29) must be equality. It then follows that equality in eqs. (23) and (24) must hold for all k (note that $x_k > 0$ and $y_k > 0$ for all k). Tracing back further, we know that equality holds for all k in eq. (21), from which condition (b) immediately follows.

(b) $\Rightarrow$ (c): Assume that there is a positive number $\bar{\sigma}$ such that

$$\bar{\sigma} = \sqrt{b_{kk}^2 + \frac{1}{4}\left(b_{k-1,k} - b_{k,k+1}\right)^2} - \frac{1}{2}\left(b_{k-1,k} + b_{k,k+1}\right) \tag{30}$$

holds for all k. Then we have

$$b_{kk}^2 = (b_{k-1,k}+\bar{\sigma})(b_{k,k+1}+\bar{\sigma}) \quad (k = 1, 2, \ldots, n). \tag{31}$$

Now, given a positive number $\bar{x}_1$, we define the sequence of positive numbers $\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_{n+1}$ to satisfy

$$\frac{b_{kk}}{b_{k,k+1}+\bar{\sigma}} = \frac{\bar{x}_{k+1}}{\bar{x}_k} \quad (k = 1, 2, \ldots, n). \tag{32}$$

Then, from eqs. (31) and (32), we have

$$\frac{b_{kk}}{b_{k-1,k}+\bar{\sigma}} = \frac{\bar{x}_k}{\bar{x}_{k+1}} \quad (k=1,2,\ldots,n). \tag{33}$$

Now we determine $\bar{x}_1$ so that $\sum_{k=1}^{n} x_k^2 = 1$ and define two n-dimensional vectors $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ by

$$\bar{\mathbf{x}} = (x_1, x_2, \ldots, x_n)^T, \tag{34}$$
$$\bar{\mathbf{y}} = (x_2, x_3, \ldots, x_{n+1})^T. \tag{35}$$

Then, using eqs. (32) and (33), it is easy to see that these vectors satisfy the following relations:

$$B\bar{\mathbf{x}} = \bar{\sigma}\bar{\mathbf{y}}, \tag{36}$$
$$B^T\bar{\mathbf{y}} = \bar{\sigma}\bar{\mathbf{x}}. \tag{37}$$

Moreover, since $\bar{\mathbf{x}}$ is normalized, we have from eqs. (36) and (37),

$$\bar{\mathbf{y}}^T\bar{\mathbf{y}} = \frac{1}{\bar{\sigma}}\bar{\mathbf{x}}^T B^T\bar{\mathbf{x}} = \bar{\mathbf{x}}^T\bar{\mathbf{x}} = 1. \tag{38}$$

Hence, $\bar{\sigma}$ is a singular value of B and $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are the corresponding normalized right and left singular vectors, respectively. Furthermore, since $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are positive vectors, from the Perron-Frobenius theory applied to $(B^TB)^{-1}$ and $(BB^T)^{-1}$, we know that $\bar{\sigma}$ is the smallest singular value. Also, $x_{n+1} = x_1$ follows from $\bar{\mathbf{x}}^T\bar{\mathbf{x}} = \bar{\mathbf{y}}^T\bar{\mathbf{y}} = 1$. Thus we have proved that condition (c) is satisfied.

(c) $\Rightarrow$ (a): Assume that condition (c) is satisfied. Then from the k-th component of $B\mathbf{x} = \sigma_n\mathbf{y}$, we have

$$b_{kk}x_k = (b_{k,k+1} + \sigma_n)y_k \quad (1 \le k \le n). \tag{39}$$

Similarly, from the k-th component of $B^T\mathbf{y} = \sigma_n\mathbf{x}$,

$$b_{kk}y_k = (b_{k-1,k} + \sigma_n)x_k \quad (1 \le k \le n). \tag{40}$$

Multiplying eqs. (39) and (40) sides by sides and noting that $x_k > 0$ and $y_k > 0$, we obtain

$$b_{kk}^2 = (b_{k,k+1} + \sigma_n)(b_{k-1,k} + \sigma_n) \quad (1 \le k \le n), \tag{41}$$

or

$$\sigma_n = \sqrt{b_{kk}^2 + \frac{1}{4}\left(b_{k-1,k} - b_{k,k+1}\right)^2} - \frac{1}{2}\left(b_{k-1,k} + b_{k,k+1}\right), \tag{42}$$

which shows that the Ostrowski-type bound gives the smallest singular value. $\square$

4 Example

We close this paper by giving an example of a bidiagonal matrix to which Theorem 3.3 can be applied.

Consider an $n \times n$ upper bidiagonal matrix

$$B = \begin{pmatrix} \sqrt{\sigma(\sigma+b)} & -b & & & \\ & \sigma+b & -b & & \\ & & \ddots & \ddots & \\ & & & \sigma+b & -b \\ & & & & \sqrt{\sigma(\sigma+b)} \end{pmatrix}, \tag{43}$$

where $\sigma > 0$, $b > 0$, all the upper subdiagonal elements are $-b$ and the second to $(n-1)$th diagonal elements are $\sigma + b$. For this matrix, the argument of the minimum in the right-hand-side of the Ostrowski-type lower bound gives the same positive value σ for all k. So the condition (b) in Theorem 3.3 is satisfied and the value σ given by the Ostrowski-type bound should be the smallest singular value.

If we define two n-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} = c\begin{pmatrix} 1 \\ \frac{\sqrt{\sigma(\sigma+b)}}{\sigma+b} \\ \vdots \\ \frac{\sqrt{\sigma(\sigma+b)}}{\sigma+b} \\ \frac{\sqrt{\sigma(\sigma+b)}}{\sigma+b} \end{pmatrix} \quad \text{and} \quad \mathbf{y} = c\begin{pmatrix} \frac{\sqrt{\sigma(\sigma+b)}}{\sigma+b} \\ \frac{\sqrt{\sigma(\sigma+b)}}{\sigma+b} \\ \vdots \\ \frac{\sqrt{\sigma(\sigma+b)}}{\sigma+b} \\ 1 \end{pmatrix}, \tag{44}$$

where c is a normalization constant, it is easy to see that $\mathbf{x}$ and $\mathbf{y}$ are singular vectors of B belonging to the singular value σ. Since both are positive vectors, it is clear that σ is the smallest singular value of B. Hence the condition (a) is satisfied. Also, condition (c) holds apparently.

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